In an environment of low public market expected returns, the higher expected returns of some illiquid private assets look enticing despite the lack of a secondary market – investors effectively must hold these illiquid assets until maturity.

Illiquid private assets are not necessarily illiquid versions of their liquid public counterparts. While some illiquid assets are quite similar to liquid public assets in terms of their risks and potential returns, other private assets, such as private equity and debt limited partnerships (LPs) have characteristics that offer investors opportunities not available in the public markets. For example, in contrast to many public assets, private LP assets are often concentrated in smaller companies, with terms negotiated with one or a few sponsors/originators who may provide close monitoring and frequent interventions with management. Consequently, these illiquid assets may offer investors a new source of alpha not available with public assets, arising from the manager’s private information and skill to select, bargain with, monitor and guide private companies. These higher expected returns constitute a “private market premium” over liquid public markets.

The higher expected returns of illiquid assets may help improve portfolio performance. However, a challenge for investors is to determine their allocation to illiquid assets subject to their cash flow obligations. Many investors are comfortable with the notion of segmenting their portfolios into a liquid component and an illiquid component. In fact, it is common for investors to define an “illiquid bucket” for the portion of their assets that will not be needed for liquidity purposes for many years. But, what is the optimal illiquid-liquid asset mix?

We present a stylized asset allocation framework to illustrate how an investor might determine their allocation to illiquid assets, subject to how confident they wish to be regarding their portfolio’s cash flow constraints. This framework highlights how the illiquid-liquid allocation has consequences for the composition of the investor’s liquid public market portfolio as a less-risky liquid portfolio may allow the investor to increase allocation to illiquid assets. Although, for simplicity, our framework
For the discussion below we chiefly consider equity and debt LP investments which have grown as an investment option for “illiquid-capable” investors such as endowments, pension and sovereign wealth funds. Furthermore, since the consequences of failing to satisfy a cash flow shock could be severe, it is not clear that those investors who can confidently determine some fraction, however small, of their portfolios that will not be needed to raise cash in the years ahead. It is not obvious how large a risk premium the market would generate for this uncertainty risk. It seems reasonable that some investors can manage this uncertainty by employing strategies that mitigate their vulnerability to liquidity shocks such as by diversifying “uncertainty” risk, investors would insist on compensation with larger expected performance from their illiquid investments. However, some investors may never be completely certain about their capacity to hold illiquid assets. For them, there is always some chance that they could suffer a liquidity shock – either an unexpected need for cash or an unexpected decline in the value of the liquidity portfolio – that would require them to make other arrangements to raise cash (e.g., borrow, default on cash commitments, postpone projects, or otherwise reduce the need for cash) which are generally undesirable. Considering this “uncertainty” risk, investors would insist on compensation with larger expected performance from their illiquid investments. However, investors can manage this uncertainty by employing strategies that mitigate their vulnerability to liquidity shocks such as by diversifying the scheduled maturities of illiquid investments and resisting the temptation to over-commit to such investments.

Liquid public assets are publicly-issued (e.g., SEC-registered) and freely tradeable among investors in secondary markets. While some public equities and bonds may not trade often, an investor can sell public assets within a reasonable period and at a modest price concession, even without a centralized exchange. Public companies tend to be relatively large companies that file regular financial reports, and whose business activities are followed by security analysts and the press. Also, public assets often belong to major performance benchmarks (e.g., S&P 500 equity index or the Bloomberg Barclays Aggregate bond index) which require daily transparent valuation. In addition, regulatory agencies often require prompt public transaction reporting. For liquid public assets, both the issuing company and investors have similar information regarding the assets’ quality. Companies, and the sponsors/underwriters of their securities, are required to divulge material information and investors are not permitted to trade on material non-public information. Terms and conditions tend to be homogenous, with few performance covenants and little negotiation with investors prior to sale. After issuance, investors rely on publicly-available information to assess the company’s performance. An individual investor has little power to influence the company’s actions and financial condition. If the investor does not like what the company is doing the only option is usually to sell and walk away. In contrast, private assets are unregistered and have minimal public reporting requirements. In addition, these assets do not have an active secondary market. Although private asset secondary transactions can occur, they are rare and can take a long time to execute. In all practicality, buyers of illiquid private assets know they must hold until maturity, sale/IPO, default, refinance or re-negotiation.

Private Market Premium

How much return premium will investors demand from illiquid private assets1 – a “private market premium” – and, as compensation for what risk? Compared to liquid public assets, the characteristics of some illiquid private assets offer a number of new potential risks and rewards which investors should consider. The expected return advantage of some illiquid private assets over liquid public assets should reflect compensation for any additional market risk and other systematic risk factors related to these illiquid private asset characteristics.

1. Structural complexity. An illiquid private asset can be structurally complex, where the underlying sources of cash flow can be hard to discern and forecast either because the ultimate sources are hidden behind a conditional waterfall structure or the success of the underlying investment itself is binary (e.g., the reorganization of a struggling business either works or does not). Since investor anxiety about complexity is likely to rise during poor market environments, illiquid private asset investors will demand compensation for complexity.

2. Rebalancing opportunity cost. Some illiquid private assets may also impose an opportunity cost on investors in the form of foregone portfolio rebalancing opportunities. In other words, an allocation to illiquid assets may mean that an investor cannot dynamically allocate across the various illiquid markets or between liquid and illiquid markets to take advantage of better return possibilities if they arise. Also, the risk attributes of the illiquid portfolio may change, and it would not be possible to rebalance the illiquid portfolio to bring it back to a desired risk level. However, the liquid portfolio remains available to make some overall portfolio expected return and risk adjustments. Nevertheless, illiquid private assets may handcuff the investor from a dynamic asset allocation perspective, therefore imposing a potential opportunity cost for which investors would insist on compensation in the form of better expected performance relative to liquid public assets.

3. Uncertainty of liquidity needs. Since investors know illiquid assets cannot be sold to raise cash, investors must be confident that they can meet their cash obligations solely from their liquid assets (i.e., the “liquidity portfolio”) over the lifetime of their illiquid investments. However, some investors may never be completely certain about their capacity to hold illiquid assets. For them, there is always some chance that they could suffer a liquidity shock – either an unexpected need for cash or an unexpected decline in the value of the liquidity portfolio – that would require them to make other arrangements to raise cash (e.g., borrow, default on cash commitments, postpone projects, or otherwise reduce the need for cash) which are generally undesirable. Considering this “uncertainty” risk, investors would insist on compensation with larger expected performance from their illiquid investments. However, investors can manage this uncertainty by employing strategies that mitigate their vulnerability to liquidity shocks such as by diversifying the scheduled maturities of illiquid investments and resisting the temptation to over-commit to such investments.

It is not obvious how large a risk premium the market would generate for this uncertainty risk. It seems reasonable that some investors can confidently determine some fraction, however small, of their portfolios that will not be needed to raise cash in the years ahead. Furthermore, since the consequences of failing to satisfy a cash flow shock could be severe, it is not clear that those investors who are

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1 For the discussion below we chiefly consider equity and debt LP investments which have grown as an investment option for “illiquid-capable” investors such as endowments, pension and sovereign wealth funds.
uncertain would be tempted by any magnitude of a risk premium to hold more illiquid assets than is prudent. Any failure to meet cash flow requirements may be an idiosyncratic investor event rather than a market systemic event, and so, would not produce a risk premium.

4. Privately negotiated terms. Unlike liquid public assets, for some illiquid private assets there is negotiation between the private company and investor regarding the asset’s terms and conditions. As such, there is scope for the investor to earn higher risk-adjusted returns from the investor’s private information, operational skill, bargaining power during negotiations and on-going oversight of the company. An investor can use private information (e.g., experience with certain companies in years past) to identify company attributes (e.g., management quality and local market conditions) that are likely to produce good asset performance. Since illiquid assets are generally held to maturity, the investor must continually use operational skill to deal with hiccups along the way. Finally, although market dynamics are changing as more private market investors enter the marketplace, a private company may have fewer financing choices compared to a large public company. A private investor may be able to negotiate more favorable terms (e.g., covenants) from this stronger bargaining position, thus increasing the potential for enhanced alpha relative to public investors who have less opportunity to exercise these skills.  

However, the ability of private asset investors to generate alpha has limits. The supply of private assets is typically less than the potential demand, so private companies can rely on competition among investors to keep the latter’s bargaining power in check. Also, when raising funds, a company will consider a range of funding sources: private equity, public equity, private loan, or, perhaps, a public loan. Many factors will influence the company’s decision, and cost (i.e., regulatory and monitoring costs) will be an important input. From a pricing perspective, a private company’s willingness to pay more to avoid the higher costs of complying with public market regulatory requirements may vanish if the private market alternative becomes too expensive and burdensome (i.e., covenants and oversight). Companies have a range of financing choices, so the private market cannot be too expensive relative to the public market.

5. Leverage or market risks. A potential benefit of some illiquid private assets is the ability to take more risk. Over time, capital markets have rewarded investors for taking more risk. For example, while the public corporate bond market has long-duration, high-risk assets outstanding, current expected returns are less than what is available from some private debt LPs. Similarly, public equity markets have an increasing scarcity of smaller, faster growing — but, potentially riskier — firms as new companies may have less need for fixed assets and can readily finance themselves using private markets. Both illiquid private debt and equity LP investments may offer higher expected returns — in return for higher risk — than are available in the public markets.

6. Valuation stability. Finally, some illiquid private assets may offer investors another important advantage: valuation stability for periodic reporting purposes. To illustrate we compare quarterly floating rate total returns for the Bloomberg Barclays High Yield (B-rated) index against quarterly IRRs (“internal rate of return”) for pooled private debt LPs. From January 2000 to March 2017, the standard deviation of the high yield index total returns was 6.1% compared to 4.6% for the debt LP IRRs. This is notable as these private LPs are usually regarded as riskier investments. During 2008:Q4, the index was down 23.8% whereas the LPs were down only 13.9%, and during 2015:Q3, the index was down 6.9% compared to +0.5% for the LPs. Given the potentially lower reported volatility of some illiquid private assets, an investor may be willing to accept somewhat lower expected returns relative to comparable-risk public assets.

Illiquid-Liquid Asset Allocation Framework – Overview

The characteristics of some illiquid private assets offer the potential for higher overall expected returns and may help improve portfolio performance. However, a challenge for investors is to determine their allocation to illiquid assets subject to their cash flow obligations. Assuming illiquid assets offer higher expected returns relative to public markets, what is the optimal illiquid-liquid asset mix?

We offer a stylistic asset allocation framework to help investors better understand the issues involved when allocating to illiquid assets. A feature of our model is that we explicitly treat illiquid assets as illiquid — they cannot be sold until maturity.

2 In addition, unlike a public asset which has a large population of investors, illiquid private assets often have only a few investors when facilitates dealing with difficulties when they arise. For example, instead of having to negotiate with a large investors’ committee, the private company can work with a single investor to adjust loan terms. This workout flexibility can reduce default probability and/or increase recovery values, increasing performance.


3 Middle market/direct lending private debt LPs. While it can be potentially misleading to compare total returns directly with IRRs, the risk of a spurious comparison in this instance is low. First, the data periods are relatively short (i.e., one quarter). Second, the IRRs are relatively modest in scale. Only a handful are greater than 10% (in absolute value), and none is greater than 21% (in absolute value). Spurious comparisons between total returns and IRRs become potentially more likely as the length of the reporting window increases and as the IRRs become large in absolute value.

Since the public bond index contains fixed rate assets while private debt funds generally contain floating rate assets, we take the reported index excess returns (i.e., total returns less the total returns for the matched-duration fixed-rate Treasury portfolio) and add 3m LIBOR returns to produce total returns for a “floating rate” version of the high yield public bond index. This helps remove the effect of the strong Treasury market rally during the data period that fixed-rate assets enjoyed. We also use all vintages for the pooled private debt LP fund returns to minimize the chance that just a few funds could skew the IRRs.

4 Source: Barclays POINT, Burgess and PGIM IAS.
There are other asset allocation models that incorporate illiquid assets. Generally, however, these models try to fit illiquid assets within a traditional, single-period asset allocation framework (e.g., mean-variance optimization). However, this framework implicitly assumes that the available assets are tradeable and portfolio holdings can be rebalanced. These asset allocation models acknowledge that the periodic valuations and returns of illiquid assets do not reflect “transactable” values, and typically “unsmooth” the return series using an autoregressive model. Often, this “unsmoothing” process makes illiquid assets look relatively attractive with their higher expected returns but with volatility comparable to liquid public assets. Mean-variance optimization then proceeds, lumping illiquid assets together with liquid assets, and generates an efficient frontier showing the best return for various levels of single-period volatility.

However, if an investor’s illiquid assets are truly not tradeable, we argue that the traditional asset allocation approach is not appropriate. Instead, a key risk the investor faces when allocating to illiquid assets is not having enough liquid assets that can be sold for sufficient net cash when needed. We believe that this risk should be explicitly recognized in the asset allocation model. In the framework presented below we assume the investor selects an asset allocation to maximize total portfolio value (or expected utility) at a horizon date (time $T$) subject to the constraint that the value of the liquid assets over the horizon is sufficient to satisfy any interim cash flow requirements at a pre-specified level of confidence. Consequently, the fundamental tradeoff for investors is between their total portfolio value at the horizon date versus how confidently they can satisfy their liquidity needs up to the horizon date.

Our asset allocation framework addresses the following questions:

- **How can an investor determine the optimal allocation between liquid and illiquid assets** based on maximizing horizon portfolio value and satisfying liquidity requirements up to the horizon date?
- How does an investor’s need for liquidity before the horizon date interact with the investor’s objective of maximizing horizon portfolio value (i.e., returns)? In other words, **what is the cost of the investor’s liquidity constraint?**
- By how much would horizon portfolio expected returns decrease if the investor wanted to be more confident in being able to meet the interim cash flow requirements (e.g., 90% certain rather than 80% certain)? In other words, **what is the marginal cost of increased liquidation certainty?**
- How does the investor’s illiquid-liquid asset allocation decision affect the asset allocation within the liquid asset portfolio?
- **How do other factors** (e.g., the investor’s horizon, the magnitude of any illiquid asset expected return advantage over liquid assets, and the sensitivity of the liquidation threshold to the economic state) impact the optimal allocation results?
- An investor may already have an allocation to illiquid assets. Given the investor’s liquidity requirements, what is the optimal asset allocation within the liquid portfolio?

We propose an asset allocation framework that is intuitive and flexible, and can be customized to accommodate different liquid and illiquid asset types and liquidity requirements. This section provides an overview of the framework.

There are three assets: one illiquid asset and two liquid assets. Of the two liquid assets, we assume that one has low-volatility (i.e., the “liquid low-risk asset”) and the other has high-volatility (“liquid high-risk asset”). We also assume, for this example, that the illiquid asset, which is not tradeable before the investor’s horizon date, has both higher risk and a greater expected return (i.e., a “private market return advantage”) compared to the liquid high-risk asset. The illiquid asset’s return advantage and risk, are parameters of the model—investors can input different assumptions. By varying these assumptions, the framework shows how the optimal mix of illiquid-liquid assets—and the optimal mix of liquid assets—changes.

We assume the investor has a liquidity requirement that the portfolio’s liquidation value must meet a specified liquidation threshold. However, the investor can choose a desired confidence level in meeting the liquidity requirement. For example, an investor might say: “I want to be 90% confident that the portfolio’s liquidation value will satisfy my liquidation threshold over my investment horizon of 10 years.” The investor’s liquidation threshold ($LT_t$) is measured in absolute dollar terms and can vary over time. In other words, the investor desires that the liquidation value of the liquid portfolio ($LV_t$), after transaction costs, is always greater than or equal to the liquidation threshold value (i.e., $LV_t \geq LT_t$ for all $t$ up to $T$).

5 For example, see: W. Kinlaw, M. Kritzman, and D. Turkington, “Liquidity and portfolio choice: a unified approach,” Journal of Portfolio Management (Winter 2013), which identifies and quantifies the benefits of liquid versus illiquid assets. They then introduce a shadow liquidity asset and liability in a traditional mean-variance optimization framework which includes illiquid assets (with their estimated returns unsmoothed). Any violation of a portfolio liquidity constraint is assumed to be covered by borrowing; and, M. J. Hayes, A. Primbs and B. Chiquoine, “A penalty cost approach to strategic asset allocation with illiquid asset classes,” Journal of Portfolio Management (Winter 2015), which assumes a liquidity score for various assets and then subtracts an overall portfolio liquidity penalty from expected returns in a standard mean-variance optimization objective function. However, there is no objective liquidity requirement.

6 Our framework is in the spirit of that developed in A. Ang, D. Papanikolau, and M. Westerfield, “Portfolio choice with illiquid assets,” Netspar Discussion Paper (August 2013). This model has an infinite-horizon investor, who is motivated to consume smoothly out of wealth over time and must choose between liquid and illiquid assets. In their framework, the one illiquid asset is tradeable, but at random times. Consequently, the investor is uncertain on when it can be available for consumption.

7 We currently model the investor’s objective as maximizing expected return, while constraining on the liquidation value staying above the liquidation threshold value at a given confidence level. The limitation of this objective function is that it does not consider the portfolio valuation risk in addition to liquidity risk so that an investor would choose their optimal asset allocation by balancing return and risk. Our model can accommodate adding a risk penalty term whose size depends on portfolio valuation risk and the investor’s level of risk aversion in the objective function. In addition, there are several ways to model the risk penalty term. One way is to represent risk penalty by portfolio variance. However, considering the possibility of significant tail risk with illiquid assets, and the fact that standard deviation would not properly capture this risk, we could model the risk penalty term using expected shortfall.
There are many possible choices for modeling the liquidation threshold, and each investor will likely have a different specification to reflect their own circumstances. We assume the liquidation threshold grows at an annual rate equal to the liquid low-risk asset’s expected annual return (i.e., some trend growth rate). Other liquidation threshold specifications are also possible. For example, an investor may have a target level of liquidity at specific future dates (e.g., a liability schedule).

An investor’s liquidation threshold could have a random component. For example, an investor’s liquidation threshold may have a normal level of background volatility or it may depend on the unpredictable state of the economy. During “bad” economic states, the liquidation threshold may suddenly increase. The framework presented below assumes that when the economy is doing well (i.e., a “good” economic state) the liquidation threshold increases at trend. However, if the economy is doing poorly (i.e., a “bad” economic state), the threshold suddenly increases, and reverts to trend only after the economy returns to a good state. How sensitive the threshold is to bad economic states is a parameter of the model.

The state of the economy can also affect the liquid portfolio’s liquidation value. Specifically, in addition to liquid portfolio returns being sensitive to the state of the economy, we also assume that transaction costs (i.e., bid-ask spread and market impact cost) for the liquid portfolio vary with the economic state. Transaction costs increase in bad economic states. In addition, we assume the total market impact cost increases with the absolute size of the liquid portfolio that needs to be sold to meet the liquidation threshold.

By making both the liquidation threshold and liquidation value conditional on the state of the economy we hope to capture some of the real challenges faced by investors: In bad economic states, when the investor’s liquidity requirements are increasing unexpectedly, the liquidity portfolio will likely be performing poorly and will be particularly costly to liquidate to raise cash.

The framework simulates both the dynamics of the asset returns and the behavior of the liquidation threshold. We base the simulation for asset returns on their long-term historical performance, including periods of poor performance: 2000-01 and 2008-09. For a given liquidation threshold and asset allocation, we measure the likelihood that the portfolio will satisfy the liquidation threshold up to the horizon date.

Key findings gleaned from the asset allocation framework include:

- As investors increase the portfolio’s liquidity requirement, either a higher liquidation threshold value or a higher confidence level, the portfolio’s allocation to liquid assets generally increases (i.e., the allocation to the illiquid asset decreases); the weight allocated to the liquid high-risk asset generally decreases relative to the liquid low-risk asset; and the portfolio’s expected return declines.
- At some high level of confidence in meeting a portfolio’s high liquidation threshold, the investor will make only a small (or, no) allocation to the illiquid asset, irrespective of the return advantage of the illiquid asset over the liquid high-risk asset.
- For a given level of confidence, as an investor increases the investment horizon the optimal allocation to the liquid portfolio declines, as does the allocation to the liquid high-risk asset. This result supports the oft-heard comment that “long-horizon” investors are better off allocating more to illiquid assets.
- As the expected return advantage of the illiquid asset increases, the investor, for a given level of confidence will decrease both the allocation to the liquid portfolio and the allocation to the liquid high-risk asset. In other words, the investor will increase the illiquid allocation to pick up additional return but will protect the portfolio’s liquidation value by increasing allocation to the liquid low-risk asset.

### Figure 1: Assumed Investment Opportunity Set: Three Assets

<table>
<thead>
<tr>
<th>Return Statistics</th>
<th>Liquid Low-Risk</th>
<th>Liquid High-Risk</th>
<th>Illiquid*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized Expected Return</td>
<td>4.80%</td>
<td>7.40%</td>
<td>12.40%</td>
</tr>
<tr>
<td>Annualized Standard Deviation</td>
<td>3.40%</td>
<td>9.20%</td>
<td>17.30%</td>
</tr>
</tbody>
</table>

* Illiquid Asset return is assumed to have an annualized return advantage of 5% over the Liquid “High Risk” asset

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Liquid Low-Risk</th>
<th>Liquid High-Risk</th>
<th>Illiquid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid Low-Risk</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Liquid High-Risk</td>
<td>-0.22</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Illiquid</td>
<td>-0.22</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Source: Barclays POINT, PGIM IAS. Example shown for illustrative purposes only.
These findings are sensitive to the severity of the investor’s liquidity requirements, both the initial liquidity threshold and the desired confidence level. Asset allocations, both illiquid vs. liquid and liquid low-risk vs. liquid high-risk, do not necessarily change monotonically with the liquidity requirements. There can be “reversals” depending on both the initial level of the liquidity requirements and the specifics on how the liquidity requirements change.

The asset allocation framework can accommodate two different rebalancing assumptions for the liquid portfolio: no rebalancing over the horizon once the initial liquid portfolio asset allocation decision has been made, or allowing periodic (monthly) rebalancing back to the initial asset allocation (including transaction costs). The results presented below assume monthly rebalancing.

**ASSET ALLOCATION FRAMEWORK - DETAIL**

**Asset choices and dynamics**

We assume the investor has a desired investment horizon date (T) and seeks to maximize expected portfolio value at the horizon date but is constrained to meet a liquidation threshold over the entire horizon. The investment opportunity set includes one illiquid asset and two liquid assets. The illiquid asset, which forms the investor’s illiquid portfolio, has relatively high expected returns. It also has a maturity equal to the investor’s horizon. We assume the illiquid asset is truly illiquid – it cannot be sold to meet the portfolio’s liquidation threshold. Although we specify an annualized monthly standard deviation of returns for the illiquid asset, this volatility only affects the potential distribution of the portfolio’s value at the horizon date when the illiquid asset can be sold.

The two liquid assets include one “high-risk” asset and one “low-risk” asset, with the high-risk asset having a higher monthly return volatility and expected return. Both liquid assets, which compose the investor’s liquid portfolio, can be rebalanced periodically and can be sold to raise cash, if needed, but will incur transactions costs to do so (i.e., bid-offer and price impact costs).

For our example, the liquid low-risk asset has the same monthly return distribution as the Bloomberg Barclays 3-5y US Treasury Bond Index and the liquid high-risk asset has the same return distribution as the Bloomberg Barclays High Yield Corporate Bond Index. Of course, other liquid asset choices could be employed. We assume the illiquid asset has a similar return distribution as that of the liquid high-risk asset, but with an expected return advantage and higher volatility. Users of the model can make different assumptions.

We assume the correlation between the two liquid assets equals the same correlation between the respective indices over the period from February 1997 to November 2016. Monthly returns for the illiquid asset and liquid high-risk asset are assumed to be perfectly correlated for simplicity.

Figure 1 summarizes the assumed annualized return statistics for the three assets in the investor’s opportunity set. These return assumptions are solely to illustrate the model, they are not forecasts of future returns and volatilities.

**Investor’s liquidity requirements**

The investor wants some level of confidence that the liquid portfolio can generate enough cash – after transaction costs – to satisfy the portfolio’s liquidation threshold. In other words, the liquid portfolio must have a liquidation value (LV) that is greater than or equal to the liquidation threshold value (LT). The illiquid asset is not available to meet any cash flow requirements before the horizon date.

At the outset, the investor specifies a required initial liquidation threshold as a percentage of the total portfolio’s initial value,

$$LT_0 = k_0 \times MV_0$$

where $k_0$ is the initial liquidation threshold value (e.g., $k_0 = 0.7$ is called a “70% liquidation threshold value”) and $MV_0$ is the initial total portfolio market value. The liquidation threshold is measured in dollar terms. The liquidation threshold, $LT_0$, could remain unchanged up to the horizon. However, we allow the threshold value to have its own dynamics to reflect the likelihood that an investor would want the liquidation threshold to grow with time and increase during poor economic states. Consequently, we assume the liquidation threshold evolves as follows:

$$LT_1 = (1 + \text{trend rate}_s)^s \times (1 + s \times \text{economic state}) \times LT_0$$

Ignoring the economic state term for the moment, the liquidation threshold follows a trend. For example, investors may wish to have the liquidation threshold grow with GDP, wage growth, or inflation. While the model allows for the trend rate to vary, for simplicity we assume the trend rate is constant and equals the expected average return for the liquid low-risk asset. We also may want the liquidation

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8 For simplicity, we assume the investor maximizes terminal total portfolio value. Other objective functions could be adopted such as maximizing (mean-variance or loss-aversion) expected utility at the horizon.

9 We assume the investor is not able to borrow against the illiquid asset.

10 Therefore, in this example, the illiquid asset might be thought of as a more levered version of the high-risk liquid asset. The correlation structure could be modified as part of a broader asset allocation framework.

11 The framework could accommodate periodic cash flows generated by the illiquid asset to be added to the liquidity portfolio over the horizon period. This might be particularly relevant for investors whose initial liquidity threshold value is greater than 100%. In the current setup, we assume the illiquid asset generates no interim cash flows.
We assume that the liquidation threshold shifts upwards from trend in bad economic states. As explained below (see “Simulation and states of the economy” section), we define a “bad” state as when the liquid high-risk asset is doing poorly. So, in a bad state, not only will the liquid portfolio be under valuation stress, but the portfolio will be experiencing higher than expected cash flow requirements. This setup aims to capture the added strain that portfolios may experience in bad economic times. We also include an “$s$” parameter that adjusts the sensitivity of the liquidation threshold to the economic state. Investors can use the framework assuming different sensitivity values. Our default value for $s$ is 1.0.

Figure 2 illustrates the dynamics of the liquidation threshold. As shown, the threshold increases along a trend, but sometimes moves above trend during bad economic states (i.e., when the liquid high-risk asset has poor returns), and then reverts to trend.

**Investor’s Asset Allocation Problem**

In our framework, the investor’s risk is that the portfolio’s liquidation value ($LV_t$) falls below the liquidation threshold ($LT_t$) in any month before the horizon date. Once the investor has addressed this risk, the investor then seeks to maximize the portfolio’s expected value at the horizon date.

The investor wants to satisfy the portfolio’s liquidity requirement with a certain level of confidence. For example, the investor may want the portfolio to satisfy the liquidation threshold 90% of the time over the horizon period. This desired confidence level (CL) is an investor choice variable.

We can formally state the investor’s task: Maximize horizon expected portfolio value ($MV_T$) subject to the constraint that the probability that the liquidation value stays greater than or equal to the liquidation threshold over the horizon period is at least CL. In other words:

Maximize $\text{Expected } MV_T$

s.t. $\Pr\{LV_t \geq LT_t\} \geq CL$ for all $t$ between $[0, T]$ “liquidity requirement”

$MV_0 = MV_0(\text{Liquid Portfolio}) + MV_0(\text{Illiquid Portfolio})$ “budget constraint”

12 The findings shown herein are derived from statistical models. Reasonable people may disagree about the appropriate model and assumptions. Models should not be relied upon to make predictions of actual future account performance.
Today, the investor must decide how to allocate between the illiquid and liquid portfolios. In addition, for the liquid portfolio the investor must decide how to allocate between the liquid high-risk and liquid low-risk assets. These two allocation decisions are intertwined. A higher allocation to the illiquid asset may raise expected returns, but also increases the probability that the portfolio may violate the liquidation threshold value too frequently. To reduce this risk, the investor may increase the weight to the low-risk liquid asset in the liquid portfolio. The illiquid-liquid allocation and the liquid portfolio high-low risk allocation decisions are solved simultaneously.

We summarize the investor’s asset allocation decisions with two parameters:

- **Allocation to liquid assets**:

  \[ \Theta_0 \] = “theta” represents the initial asset allocation weight of the liquid portfolio, expressed as a percentage of the overall portfolio’s initial value. At any time, \( t \), the fraction of the total portfolio’s market value that is liquid equals \( \Theta_t \); and

- **Allocation to high-risk liquid assets**:

  \[ \Phi_0 \] = “phi” represents the initial asset allocation weight of the liquid high-risk asset, expressed as a percentage of the liquid portfolio’s initial value. At any time, \( t \), the fraction of the liquid portfolio’s market value invested in the liquid high-risk asset is \( \Phi_t \).

To arrive at the optimal initial asset allocation solution the investor provides the following inputs:

- \( k_0 \) = initial liquidation threshold value;
- \( CL \) = confidence level in meeting portfolio’s liquidity requirement over the horizon;
- trend rate = trend growth rate of the liquidation threshold;
- economic state = how the liquidation threshold responds to the economic state variable; and
- \( s \) = sensitivity of the liquidation threshold to the economic state variable.

The optimal \( \{ \Theta_0^*, \Phi_0^* \} \) initial asset allocation pair (i.e., % allocation to liquid, % allocation to high-risk liquid relative to total liquid) is labeled with asterisks. Due to market movements, the value of \( \Theta_t \) will fluctuate. However, since the illiquid asset is illiquid the investor

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**Figure 3: Assumed Liquid Portfolio Transaction Costs**

**Bid-Ask Spread (as a percentage of asset’s value) in Good/Bad Economic State**

<table>
<thead>
<tr>
<th>Economic State</th>
<th>Liquid Low-Risk</th>
<th>Liquid High-Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>0.61%</td>
<td>1.01%</td>
</tr>
<tr>
<td>Bad</td>
<td>0.99%</td>
<td>2.28%</td>
</tr>
</tbody>
</table>

Source: Barclays Capital, PGIM IAS.

**Market Impact Cost (cost per $1m traded) in Good/Bad Economic State**

<table>
<thead>
<tr>
<th>Size of Trade</th>
<th>Bad Economic State</th>
<th>Good Economic State</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Liquid Low-Risk</td>
<td>Liquid High-Risk</td>
</tr>
<tr>
<td>([0, $2mm])</td>
<td>0.199%</td>
<td>0.399%</td>
</tr>
<tr>
<td>(($2mm, $3mm])</td>
<td>0.130%</td>
<td>0.293%</td>
</tr>
<tr>
<td>(($3mm, $5mm])</td>
<td>0.092%</td>
<td>0.162%</td>
</tr>
<tr>
<td>(($5mm, $15mm])</td>
<td>0.043%</td>
<td>0.081%</td>
</tr>
<tr>
<td>(($15mm, $35mm])</td>
<td>0.021%</td>
<td>0.039%</td>
</tr>
<tr>
<td>(($35mm, ))</td>
<td>0.009%</td>
<td>0.020%</td>
</tr>
</tbody>
</table>

Source: Barclays Capital, PGIM IAS. Example shown for illustrative purposes only.

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13 We permit only non-negative asset allocation weights.
14 Barclays US short duration IG bid-ask spread (LCS) and Price Impact Cost (PIM) data are used for Liquid Low-Risk asset’s bid-ask spread (LCS) and Price Impact Cost (PIM). Barclays Corporate HY bid-ask spread (LCS) and Price Impact Cost (PIM) are used for Liquid High-Risk asset’s bid-ask spread (LCS) and Price Impact Cost (PIM).
cannot take any action to rebalance \( \Theta \). The value of \( \Phi \) will also fluctuate, but since these are liquid assets the investor can choose whether or not to rebalance \( \Phi \) back to \( \Phi^* \). The model can accommodate either choice. For the example below, we assume the investor rebalances \( \Phi \) back to \( \Phi^* \) each month. We use the liquid assets' bid-ask spread to penalize the liquid portfolio's market value to cover rebalancing costs. Since the monthly rebalancing amounts are relatively small, we do not include any market impact costs.

### Simulation and states of the economy

We use simulation to solve the investor’s asset allocation problem. We assume the monthly returns for the three assets follow a joint t-distribution.\(^{15}\) For our example, the assumed asset distribution parameters are shown in Figure 1. Each month from today (\( t = 0 \)) to the horizon (\( t = T \)), we sample, with replacement, from this joint t-distribution. For example, if \( T = 10 \) years, we generate 120 monthly returns for all assets. Each 120-month time series for all assets is a single simulation run.

The framework allows for “bad” and “good” economic states which can affect both the liquidation threshold and the cost (bid-ask spread and market impact) of liquidating the liquid portfolio. Figure 3 presents our assumed transaction cost conditional on the economic state.

To determine whether a month is in a bad economic state, we first calculate the 6-month moving average (backward-looking) cumulative total return series for the liquid high-risk asset. We define the start of a bad state whenever the monthly moving average return experiences a drawdown of more than -3%. The return to a good economic state follows whenever the moving average return to a good economic state follows whenever the moving average return experiences a drawdown of less than -3%.

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15 Currently we model assets’ return using a t-distribution. Although a t-distribution has fatter tails than a normal distribution, it probably would still not be able to capture the tail events of illiquid assets. One possible enhancement is to model asset returns using a jump-diffusion-type model which essentially adds a Poisson random variable to the original asset data generating process.

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drawdown is less (in absolute value) than -3%. For example, suppose in month K the 6m moving average experiences a 3.1% drawdown from its prior peak in month A (assuming months follow alphabetical order). Month K is the start of the bad economy. Then, suppose the drawdown in month S, for the first time since month K, is less than -3% (i.e., -2.8%). We define month R as the last month of the bad economy.

The framework could accommodate many other possible definitions of bad and good economic states. For example, some investor portfolio’s cash flows may be sensitive to commodity prices, not necessarily to investment returns. Other investors may be sensitive to employment dynamics in their industry.

For each simulation run, we select an initial \( \{ \Theta_0, \Phi_0 \} \) pair\(^{16}\) and then let the simulation run from time \( t = 0 \) to \( t = T \). We check if the percentage of months over the investment horizon that the liquid portfolio’s value falls below the liquidation threshold is more than \( 1 – \text{investor's confidence level} \) (e.g., 10% if the investor sets a 90% confidence level). If so, the run is considered a liquidation threshold “failure”. For example, suppose the investment horizon is 120 months and the investor’s desired confidence level is 90%. If a simulation run produces more than 6 months wherein \( LV_t < LT \), then that run is labeled a failure.

We then run the simulation again, using the same initial \( \{ \Theta_0, \Phi_0 \} \) pair, to see if that particular simulation run is a liquidation threshold failure, and so on. Figure 4 shows an example of a “failure” (top panel) and a “success” (bottom panel) simulation run. In the top panel, \( LV_t \) is below \( LT \) for 53 of the 120 months, or 44.2% of the months. Given an assumed confidence level of 90%, this run is considered a “success.” In contrast, in the bottom panel, in no months is \( LV_t \) below \( LT \), so the run is a “success.”

For each initial \( \{ \Theta_0, \Phi_0 \} \) pair we conduct 10,000 simulation runs and calculate the percentage of runs that are “failures”.\(^{17}\) If, across all the runs the failure percentage is less than \( 1 – \text{investor's confidence level} \), we compute the average horizon portfolio market value over all the runs (including the failure). If, on the other hand, the failure percentage exceeds \( 1 – \text{investor’s confidence level} \) then we consider the initial \( \{ \Theta_0, \Phi_0 \} \) pair as an infeasible asset allocation pair.

We repeat this simulation exercise for many initial \( \{ \Theta_0, \Phi_0 \} \) pairs (increment \( \Theta_0 \) by 1% from 0% to 100%). The \( \{ \Theta_0, \Phi_0 \} \) pair that maximizes horizon average portfolio value is the optimal asset allocation solution and we label it \( \{ \Theta_*, \Phi_* \} \).

**OPTIMAL ASSET ALLOCATION RESULTS**

We generate optimal illiquid-liquid asset allocations across various initial liquidation threshold values (i.e., 20%, 50%, 70%, 80% and 90%) and desired confidence levels (i.e., 60%, 70%, 80%, and 90%) in meeting the liquidation threshold.

Figure 5 shows the optimal \( \{ \Theta_*, \Phi_* \} \) pair for various liquidation thresholds and confidence levels. For example, if the investor has a 70% liquidation threshold and a 70% confidence level (see box outlined in red in Figure 5), the optimal portfolio would have a 23% initial allocation to the illiquid asset (0.23 = 1 – 0.77) and a 100% allocation to the high-risk asset within the liquid portfolio.\(^{18}\)

**Figure 5: Optimal Initial Asset Allocation \( \{ \Theta_*, \Phi_* \} \) Pairs for Various Liquidation Thresholds and Confidence Levels**

(horizon = 10y; and illiquid asset return advantage vs. liquid high-risk asset = 5%/y)

<table>
<thead>
<tr>
<th>Liquidation Threshold</th>
<th>60% Conf. Level</th>
<th>70% Conf. Level</th>
<th>80% Conf. Level</th>
<th>90% Conf. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>(21%, 100%)</td>
<td>(22%, 100%)</td>
<td>(22%, 45%)</td>
<td>(22%, 14%)</td>
</tr>
<tr>
<td>50%</td>
<td>(51%, 100%)</td>
<td>(55%, 100%)</td>
<td>(61%, 98%)</td>
<td>(56%, 27%)</td>
</tr>
<tr>
<td>70%</td>
<td>(71%, 100%)</td>
<td>(77%, 100%)</td>
<td>(86%, 100%)</td>
<td>(82%, 44%)</td>
</tr>
<tr>
<td>80%</td>
<td>(81%, 100%)</td>
<td>(88%, 100%)</td>
<td>(98%, 100%)</td>
<td>(95%, 48%)</td>
</tr>
<tr>
<td>90%</td>
<td>(91%, 100%)</td>
<td>(99%, 100%)</td>
<td>(100%, 52%)</td>
<td>(100%, 23%)</td>
</tr>
</tbody>
</table>

Note: \( \Theta_* \) is the optimal percentage allocation to liquid assets in the total portfolio, so \( 1 – \Theta_* \) is the optimal allocation to the illiquid asset. \( \Phi_* \) is the optimal percentage allocation to the high-risk asset as a percentage of the liquid portfolio. Consequently, the total portfolio allocation to the liquid high-risk asset is \( \Theta_* \times \Phi_* \), and the total portfolio allocation to the liquid low-risk asset is \( 1 – \Theta_* \times (1 – \Phi_*) \). Please refer to Figure 1 for the assumed risk and return properties of the liquid and illiquid assets used in this example.

Source: PGIM IAS. Example shown for illustrative purposes only.

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16 Each \( \{ \Theta_0, \Phi_0 \} \) pair represents an initial allocation to liquid assets (\( \Theta_0 \)) and an initial allocation to high-risk, relative to total, liquid assets (\( \Phi_0 \)).
17 We set the number of simulation runs so that we can be 95% confident that the sample annualized return average would be 0.2% away from the true population mean.
18 For context, large public pension plans have allocations to illiquid assets (including relatively liquid hedge funds) averaging approximately 24%, with a range of 13% to 43%.
Assuming the liquid portfolio is rebalanced each month during the investment horizon to the initial liquid high-risk and low-risk relative weights, we analyze the sensitivity of the optimal asset allocation \( \{ \Theta_0^*, \Phi_0^* \} \) pairs to changes in the following model parameters:

- Investor’s confidence level \( (CL) \) in satisfying the liquidation threshold;
- Investor’s initial liquidation threshold value \( (k_0) \);
- Sensitivity of the liquidation threshold to the state of the economy \( (s) \);
- Investor’s investment horizon \( (T) \); and
- Illiquid asset’s return advantage over the liquid high-risk asset.

We will use Figure 5 to examine how the optimal \( \{ \Theta_0^*, \Phi_0^* \} \) pair varies as the investor’s liquidity requirements change. Note that the results in Figure 5 assume a 10y horizon and a 5%/y return advantage for the illiquid asset over the liquid high-risk asset. Figure 6 uses the data from Figure 5 to summarize graphically asset allocation dynamics as the liquidity requirements change.

**Asset allocation sensitivity to the confidence level and liquidation threshold**

The investor’s liquidity requirement has two components: the liquidation threshold and the desired confidence level in meeting the liquidation threshold. Changing each component has different implications for the optimal portfolio allocation. The allocation to the illiquid asset is generally less sensitive to changes in the confidence level compared to changes in the liquidation threshold.

For modest levels of confidence (say, 70% or less) the optimal percentage allocation to the illiquid asset is in the neighborhood of the value \( (1 – \text{liquidation threshold}) \). In other words, if the liquidation threshold is 70%, the portfolio will hold approximately 30% in the illiquid asset (or, 70% in the liquid assets). This is because the variability in the liquid portfolio’s returns is sufficiently low enough (given the particular liquid assets assumed in this example) to allow the liquid portfolio to meet the liquidation threshold at the target confidence level. Also, for relatively modest levels of confidence the optimal allocation (as a percentage of the liquid portfolio) to the liquid high-risk asset generally equals 100%. Although this allocation makes the liquid portfolio relatively volatile, the modest target confidence level can still be satisfied by the overall magnitude of the allocation to the liquid portfolio.

However, at higher confidence levels, for a given liquidation threshold, the investor must make the liquid portfolio’s liquidation value less risky. This can be accomplished by some combination of either increasing the allocation to the liquid portfolio or decreasing the risk of the liquid portfolio. For example, with an initial liquidation threshold of 70%, increasing the confidence level from 70% to 80%...

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**Figure 6: Optimal Initial Asset Allocation \( \{ \Theta_0^*, \Phi_0^* \} \) Pairs**

(Horizon = 10y; and illiquid asset return advantage = 5%/y)

![Graph showing optimal asset allocation for different confidence levels](image)

**Note:** \( \Theta_0^* \) is the optimal percentage allocation to liquid assets in the total portfolio, so \( 1 - \Theta_0^* \) is the optimal allocation to the illiquid asset. \( \Phi_0^* \) is the optimal percentage allocation to the liquid high-risk asset as a percentage of the liquid portfolio. The optimal \( \{ \Theta_0^*, \Phi_0^* \} \) data are from Figure 5. Horizon returns are annualized geometric average total returns.

Source: PGIM IAS. Example shown for illustrative purposes only.

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19 However, for low levels of confidence, say 20%, the allocation to the illiquid asset is generally much larger than \( 1 - \text{liquidation threshold} \). For example, with a 20% confidence level and a 70% liquidation threshold, the optimal allocation to the illiquid asset is 54%. With such a low confidence level, the portfolio can tolerate a higher allocation to the illiquid asset.
increases the allocation to liquid assets from 77% to 86%. Note that the allocation to the liquid high-risk asset, as a percentage of the liquid portfolio, remains unchanged at 100%. The overall size of the liquid portfolio is now larger, providing a sufficient buffer to its liquidation value to satisfy the higher confidence level.

These changes in portfolio allocation are intuitive. From the perspective of maximizing horizon returns, increasing the allocation to the liquid portfolio to satisfy a higher confidence level is costly given the assumed return advantage of the illiquid asset over the liquid high-risk asset. A less costly way to satisfy the higher confidence level and maintain returns while reducing the allocation to the illiquid asset is to simultaneously increase the overall portfolio’s holding of the liquid high-risk asset. These actions generate a larger buffer in the liquid portfolio’s liquidation value so as to meet the liquidation threshold with greater confidence. However, this tradeoff between the illiquid asset and the liquid high-risk asset in the face of higher confidence levels is not necessarily monotonic. There can be “reversals” if the liquidity demands were to become very strict and the portfolio were to hit allocation boundaries.

At high confidence levels, when the optimal allocation to the illiquid asset is already small, a further increase in the confidence level can only be met by reducing the allocation to the liquid high-risk asset (hence, increasing the allocation to the liquid low-risk asset). However, this produces a significant giveup in horizon expected returns. Given the relative expected returns among the assets, we may see a somewhat counter-intuitive allocation reversal. In other words, as the liquid portfolio becomes less risky by increasing the allocation to the liquid low-risk asset, the optimal solution may then lead to a modest increase in the illiquid asset. For example, increasing the confidence level from 80% to 90% (same 70% liquidation threshold) produces an optimal 4 percentage point increase in the allocation to the illiquid asset (from 14% to 18%) while at the same time the allocation to the liquid high-risk asset falls from 100% to 44%. However, once the target level of confidence becomes very strict, the portfolio has little choice but to reduce the allocation to the liquid high-risk asset as well as to reduce the allocation to the illiquid asset.

So far, the discussion of asset allocation sensitivity to changes in the confidence level was for a 70% liquidation threshold. Figures 5 and 6 also show that this sensitivity also depends on the level of the liquidation threshold. At low liquidation thresholds, an increase in the confidence level produces relatively small changes in the allocation to the illiquid asset. In other words, when the liquidation threshold is low, a higher confidence level can be met by sacrificing the allocation to the liquid high-risk asset while maintaining the allocation to the illiquid asset which has a greater expected return. Of course, the strength of this desire to maintain the allocation to the illiquid asset would change as the illiquid asset’s return advantage changes relative to the liquid high-risk asset.

In contrast, at high liquidation thresholds, the portfolio is constrained how it can optimally respond to higher confidence levels. To meet modestly higher confidence levels, the portfolio has little choice but to increase the size of the liquid portfolio while trying to keep a high allocation to the liquid high-risk asset. However, if the confidence level becomes very high at high liquidation thresholds, the allocation to the liquid high-risk asset must ultimately be reduced as well.

In general, as the liquidation threshold increases the investor effectively has no choice but to increase the size of the liquid portfolio (and, hence, decrease the allocation to the illiquid asset). To make the increase to the liquid portfolio as less costly as possible, the
investor endeavors to add allocation to the liquid high-risk asset (if the allocation is not already at 100%). Consequently, as the liquidation threshold increases (with a constant confidence level) we might expect the allocation to the illiquid asset to fall pari passu with an increased allocation to the liquid high-risk asset. However, the availability of this trade off to help maintain portfolio returns begins to disappear at high liquidation threshold levels. At high thresholds, the optimal portfolio is already 100% allocated to the liquid portfolio and the only way to reduce its risk further is to reduce the allocation to the liquid high-risk asset. Changes in the allocation to the liquid high-risk asset can be relatively large when there are small changes to the liquidity requirements from an already high level.

Overall, Figure 5 shows that as the portfolio’s liquidity requirements increase, either an increase in the liquidation threshold or the confidence level, the optimal allocation to the illiquid asset decreases. This is intuitive. However, the magnitude of the change in the allocation to the illiquid asset depends on which dimension of the liquidity requirement is changing and on the starting level of the portfolio’s liquidity requirement.20 Also, we see that the composition of the liquid portfolio can become less, or more, risky as the severity of the level liquidity requirement increases.

**Portfolio performance sensitivity to the confidence level and liquidation threshold**

Figure 6 shows the horizon expected return associated with the optimal initial \( \{ \Theta_0^*, \Phi_0^* \} \) pair as a function of the liquidation threshold and confidence level. As the investor’s liquidation threshold increases (i.e., moving from the left panel to the right panel), the horizon total return curve shifts downward. Similarly, as the confidence level increases (moving along the right in any single panel), the horizon total return also decreases.

For a 20% threshold (left panel), changing the confidence level has only a small effect on portfolio returns. The annualized horizon return is 9.66% at an 60% confidence level and falls only to 9.53% at a 90% confidence level. This is because the allocation to the illiquid asset is large and does not vary much as the confidence level changes. The decline in the allocation to the liquid high-risk asset at high confidence levels has a relatively minor effect on horizon returns because the illiquid allocation dominates the portfolio. In stark contrast, for a 90% threshold (right panel), increasing the confidence level dramatically reduces returns, falling from 7.24% at a 60% confidence level to 5.38% at a 90% confidence level. For a high threshold there is a large allocation to liquid assets, so the decline in the liquid high-risk asset as the confidence level increases has a relatively large negative effect on horizon returns.

Figure 6 frames the risk–return tradeoff faced by many investors: “What does it cost me if I want to be more confident about meeting my liquidity requirements?” As Figure 6 illustrates, the answer depends on the level of the liquidation threshold. If the liquidation threshold is low, then the cost to be more confident is low. In contrast, with a relatively high liquidation threshold, the cost to be more confident can be relatively high.

**Figure 8: “What is the Cost of Changing My Confidence Level?”**

Histogram of Annualized Horizon Total Returns for 80% and 90% Confidence Levels (horizon = 10y; 70% liquidation threshold; illiquid asset return advantage = 5%/y)

Note: Assuming a 10y horizon, 70% confidence level and 5% illiquid return advantage, we generate histograms for the 10,000 annualized horizon returns for optimal initial \( \{ \Theta_0^*, \Phi_0^* \} \) pairs when the confidence level is either 80% or 90%. These histograms show the relative frequency of the annualized horizon return across all 10,000 simulation runs. Source: PGIM IAS. Example shown for illustrative purposes only.

20 This is also dependent on the assets’ return, volatility, and correlation assumptions employed.
A clear way to see the effect of the liquidation threshold and confidence level on the portfolio’s performance is to examine the histogram of horizon returns. Figure 7 shows the histogram of annualized horizon returns for liquidation thresholds of 70% and 90%, both with a confidence level of 80%. As we have discussed, an increase in the liquidation threshold produces a relatively large reduction in the allocation to the illiquid asset and a large increase in the allocation (of the overall portfolio) to the liquid high-risk asset. Given the assumed large relative return advantage (i.e., 5%/y) of the illiquid asset, the histogram for the 90% threshold shifts to the left relative to the histogram for the 70% liquidation threshold.

Further, given the assumed higher standard deviation of the illiquid asset, the lower liquidation threshold also has a higher standard deviation of horizon returns.

Liquidation threshold’s sensitivity to the economy

Another important parameter of our framework is the sensitivity of the liquidation threshold to the state of the economy. We assume the liquidation threshold value increases at a trend annual growth rate equal to the liquid low-risk asset’s expected annual return. In addition, the liquidation threshold may deviate from this trend in bad economic states (see Figure 2).

We can examine how the asset allocation changes by varying the liquidation threshold’s sensitivity to the economic state. Specifically, Figure 9 shows three different values for the sensitivity parameter: 0.4, 1.0 (base case) and 2.5, assuming a 70% liquidation threshold.

Intuitively, to the extent that bad economic states are correlated with poor liquid portfolio performance, then the effect of increasing the value of the sensitivity parameter will be similar, but perhaps more muted, to an outright increase in the liquidation threshold examined earlier (Figure 7). As the economic sensitivity increases, the optimal allocation to the liquid portfolio ($\Theta^*_0$) increases across all

21 Based on the discussion in the first part of the paper, the higher expected returns from illiquid private assets may not be due solely to higher risk relative to liquid assets.
22 For example, a sensitivity parameter of 2.5 means that if the economy is in a bad state and the liquid high-risk asset has a total return of -2% one month, the liquidation threshold will increase in that month by $2.5 \times 2\% = 5\%$ on top of the pre-set liquidation threshold trend growth rate. Once the economy returns to a good state, the liquidation threshold returns to its trend growth path.
confidence levels (Figure 9, left panel). Figure 9 shows that the effect of increasing the sensitivity parameter depends on the confidence level. Not surprisingly, at low confidence levels, the sensitivity parameter has little effect on the allocation to the illiquid asset. However, at high confidence levels, we see that for a low sensitivity value the optimal allocation to the liquid assets decreases a bit (i.e., a “reversal”) along with a large reduction in the allocation to the liquid high-risk asset (Figure 9, middle panel). In contrast for the high sensitivity value the liquid asset allocation continues to increase along with a significant reduction in the liquid high-risk asset within the liquid portfolio. Finally, we see (Figure 9, right panel) that increased economic sensitivity moderately lowers horizon returns, at least at higher confidence levels.

The relatively small change in the illiquid asset allocation is a result of our specification of the liquidation threshold’s dependency on the bad economic state. Although we specify a bad economic state as occurring when the liquidity portfolio is performing poorly, and we allow the sensitivity parameter to increase 6-fold (from 0.4 to 2.5), the potential deviation of the liquidation threshold from the baseline trend is still somewhat small. Other investors may choose a sharper deviation of the liquidation threshold from the baseline trend in bad economic states.

**Sensitivity to the investor’s horizon**

How does a change in the length of the investor’s investment horizon affect the optimal asset allocation results? We compare the optimal initial \( \Theta_0^*, \Phi_0^* \) pairs for three horizons: 5y, 10y, and 15y (Figure 10). For this analysis, we keep the annual illiquid asset return advantage over the liquid high risk asset at 5%/y, the liquidation threshold sensitivity to bad economic states at 1.0, and the liquidation threshold value at 70% of initial portfolio value. As we will see, the different investment horizons have different asset allocation responses to changes in the confidence level.

Intuitively, the longer horizon would allow for more liquidity risk taking and a higher allocation to the illiquid asset and, perhaps, to the liquid high-risk asset. The liquidation threshold is an initial dollar amount that grows at a rate equal to the annualized expected return of the **liquid low-risk asset**. As time passes, the expected value of the overall liquid portfolio also grows, but faster than the growth rate of the liquidation threshold (due to some allocation to the liquid high-risk asset). A longer investment horizon means the liquid portfolio’s liquidation value has more time to grow at a faster rate than the liquidation threshold (again, using our current assumptions). This allows an investor with a longer investment horizon, for a given confidence level, to take more risk in satisfying the liquidity requirement as the investor has more latter years in which the portfolio’s liquidation value will be greater than the threshold. An investor with a short investment horizon is more constrained as the liquid portfolio has less time to grow relative to the threshold. We might expect different asset allocations as the time horizon varies, especially at higher confidence levels.

**Figure 10: Impact of Investor’s Horizon (5y, 10y & 15y) on Optimal \( \Theta_0^*, \Phi_0^* \) Pairs and Returns**

(5%/y illiquid asset return advantage; 70% liquidation threshold)

Note: \( \Theta_0^* \) is the optimal percentage allocation to liquid assets in the total portfolio, so \( 1 - \Theta_0^* \) is the optimal allocation to the illiquid asset. \( \Phi_0^* \) is the optimal percentage allocation to the liquid high-risk asset as a percentage of the liquid portfolio. Given the different investment horizons, we report annualized arithmetic average total returns rather than geometric returns.

Source: PGIM IAS. Example shown for illustrative purposes only.

For this analysis we assume the available illiquid asset matches the investor’s investment horizon.
Figure 10 presents, for three investment horizons (5y, 10y, and 15y), the optimal initial allocation to liquid assets (left panel), to high-risk as a percentage of liquid assets (middle panel), and portfolio returns (right panel). At a 60% confidence level, the longer horizons tend to have a slightly lower allocation to the liquid assets. The shortest horizon has a 72% allocation to liquid assets while the longest horizon has a 70% allocation. Within the liquid portfolio (middle panel), all three horizons have a 100% allocation to the liquid high-risk asset. This is because the liquidity requirement is not particularly strict.

For the shortest investment horizon, however, as the confidence level rises, the allocation to liquid assets increases, with a very slight decline in high-risk as a percentage of liquid. At a 90% confidence level, the shortest-horizon investor has only a 6% allocation to the illiquid asset (94% liquid) and a 98% allocation to high-risk as a percentage of liquid assets. Increasing the confidence level has led to a 22-percentage point reduction in the holding of the illiquid asset, even though the liquidation threshold remains unchanged at 70%.

As mentioned, for a (low) 60% confidence level the longest investment horizon (15y) investor has a 70% allocation to liquid assets and a 100% allocation to the high-risk within liquid assets. At a (high) 90% confidence level the cost benefit of the long horizon becomes clear: the required allocation to liquid assets reverses and falls to 79%, allowing the illiquid asset allocation to rise to 21%, considerably larger than the 6% allocation for the shortest horizon. To satisfy the higher confidence level, the 15y-horizon investor has only a 24% allocation to high-risk within liquid assets. Unlike the short-horizon investor, the long-horizon investor can tolerate a higher allocation to the illiquid asset while reducing the liquid high-risk allocation. From an expected return perspective, it is less costly to reduce the liquid high-risk allocation than reduce the allocation to the illiquid asset.

The effect of the investment horizon on portfolio horizon returns is relatively straightforward (Figure 10, right panel). The horizon extension relaxes the liquidity requirements which permits a greater allocation to the higher-returning illiquid and liquid high-risk assets. At an 80% confidence level, an extension of the investment horizon from 5y to 15y allows expected portfolio (arithmetic) average horizon total returns to increase 380bp/y, from 8.8% to 12.6%, under the assumptions employed. In practice, the investor’s investment horizon is likely not under the investor’s control as there could be uncertainty regarding how the liquidation threshold will evolve in later years. However, the analysis can help investors quantify how much portfolio returns are affected by the horizon decision, which might be a consideration in the portfolio’s performance evaluation.

**Sensitivity of the illiquid asset’s expected return advantage**

What are the ramifications of changing the illiquid asset’s return advantage over the liquid high-risk asset? Figure 11 shows what happens as the expected return advantage of the illiquid asset over the liquid high-risk asset changes. (We assume a 10y investment horizon, a liquidation threshold value of 70%, and the liquidation threshold’s sensitivity to a bad economic state to be set at 1.0. We leave unchanged the expected return difference between the liquid low-risk and liquid high-risk assets.) Given a 60% confidence level, as the annual return advantage of the illiquid asset increases from 5% to 8% per year, the allocation to liquid assets ($\theta_0^*$) is essentially...
unchanged at 71% (Figure 11, left panel) and the allocation to the liquid high-risk asset, as a percentage of liquid, remains at 100% (Figure 11, center panel). Given any illiquid return advantage, such a low confidence level allows the investor to keep the same allocations and will maximize horizon expected returns.

However, allocations change significantly as the confidence level rises. At 80%, the allocation to liquid assets rises to 86% for an illiquid return advantage of either 5% or 2%. This is an increase of 15 percentage points relative to the lower confidence level of 60%.

However, the allocation to the liquid high-risk asset remains at 100% of liquid. In other words, if the return advantage of the illiquid asset is too low, the investor will satisfy the higher confidence level by simply increasing the size of the liquid portfolio, which is fully invested in the liquid high-risk asset. In contrast, at the highest illiquid return advantage considered (i.e., 8%), the allocation to the illiquid asset is less sensitive to changes in the confidence level. Instead, as the confidence level rises, the allocation to the liquid high-risk asset falls significantly. As would be expected, different levels of the illiquid return advantage produces different asset allocation responses as the confidence level changes.

Finally, for a given confidence level, as the annual return advantage of the illiquid asset increases, the annual portfolio horizon return increases (Figure 11, right panel). This framework can be used to assess the tradeoff between the magnitude of the illiquid asset’s return advantage and an investor’s desired confidence level. For example, if an investor required a certain portfolio horizon return at a 60% confidence level, but now needs to be 90% confident, the illiquid asset’s return advantage needs to increase from 5% to 8% to maintain the same horizon expected return (as shown by the horizontal grey dashed line).

**Portfolio Dynamics**

So far, we have illustrated some of the framework’s comparative statics: how do the optimal initial \( \{ \Theta_0^*, \Phi_0^* \} \) allocation pair and portfolio horizon return change as we vary some of the model’s parameters while holding the other parameters unchanged? Investors, however, may also be interested in the behavior of the portfolio throughout the horizon period.

For example, how does the portfolio’s liquidation value \( LV_t \) evolve in relation to the liquidation threshold \( LT_t \). Unless the investor wishes to be 100% confident, there will be months when the liquidation value will be below the threshold \( i.e., LV_t < LT_t \). An investor will want to know:

- How much might the liquidation value fall short of the liquidation threshold in a given month over the investment horizon?;
- How often might the liquidation value be greater than the liquidation threshold?; and
- How does the portfolio’s allocation to the illiquid asset change over time?

![Figure 12: Distribution of MIN(LV/LT,)](10y horizon; illiquid asset return advantage = 5%/y; 70% liquidation threshold; 90% confidence level)

Note: For each simulation run, for a given optimal initial \( \{ \Theta_0^*, \Phi_0^* \} \) pair, we record the lowest value of \( LV_t/LT_t \) during the entire investment horizon. We label this value \( \text{MIN}(LV_t/LT_t) \). This histogram shows the relative frequency of the \( \text{MIN}(LV_t/LT_t) \) value across all 10,000 simulation runs.

Source: PGIM IAS. Example shown for illustrative purposes only.
Distribution of the minimum ratio of Liquidation Value over Liquidation Threshold

Recall, for each initial \(\{\Theta_0, \Phi_0\}\) pair we conduct 10,000 simulation runs. In the examples below, we assume a 10y horizon so each simulation run comprises 120 months. For each simulation run, we track \(LVT_t\) and \(LT_t\) and the ratio \(LVT_t/\text{LT}_t\) for each of its 120 months. We then note the \(\text{MIN}[LVT_t/\text{LT}_t]\) across all 120 months (i.e., how low did the “liquidity ratio” get?), for each run, and then construct a histogram of these \(\text{MIN}[LVT_t/\text{LT}_t]\) values across all 10,000 runs. Figure 12 shows results assuming a 10y horizon, 70% liquidation threshold and a 90% confidence level.

The average \(\text{MIN}[LVT_t/\text{LT}_t]\) value across all 10,000 runs is 1.04. The \(\text{MIN}[LVT_t/\text{LT}_t]\) value fell below 0.85 in only 1.89% of the simulation runs, and the lowest \(\text{MIN}[LVT_t/\text{LT}_t]\) value was 0.69. Given the assumed high confidence level, it is perhaps not surprising that the liquidation value never falls much below the liquidation threshold.

How often is \(LVT_t\) greater than \(\text{LT}_t\) over the investment horizon?

How often might the liquidation value be greater than the liquidation threshold over the entire investment horizon? This frequency can give the investor an idea of how much liquidity “buffer” the liquid portfolio is providing over the investment horizon. To answer, for each simulation run we compute the number of months over the investment horizon that \(LVT_t > \text{LT}_t\).

Assuming a 70% liquidation threshold and a 90% confidence level, 2.9% of the 10,000 simulation runs (or, 286 runs) had \(LVT_t > \text{LT}_t\) for only up to 70 months out of the 120 months (see Figure 13). In other words, in 286 simulation runs the liquidation value was less than the liquidation threshold for 50 or more months over the 120 month investment horizon. This level of occurrence is not unexpected as the confidence level is below 100%. In contrast, in approximately 72% of the simulation runs, the liquidation value was greater than the liquidation threshold in either 119 or 120 months over the 120 investment horizon.24

Portfolio allocation dynamics

Once the initial allocation to the illiquid asset is set, how does the portfolio’s illiquid-liquid allocation fluctuate over time? The weight of illiquid asset in the portfolio will change as the illiquid asset’s performance varies from the liquid portfolio’s performance.

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24 Since Figure 13 assumes a 90% confidence level and a 10y investment horizon, then we should be able to confirm that at least 90% of the 10,000 simulation runs had at least 90% of 120 months (or, 108 months) where the liquidation value is greater than the liquidation threshold (i.e., \(LVT_t/\text{LT}_t > 1\)). The number of runs of 108 months or more was 9.9% of 10,000 runs.
Figure 14: Average $\Theta_t$ and $\Phi_t$ Dynamics over the Investment Horizon
(10y horizon; 70% liquidation threshold; 90% confidence level; 5% illiquid return advantage)

Note: For a given optimal initial $(\Theta_0^*, \Phi_0^*)$ pair, we record the value of $\Theta_t$ and $\Phi_t$ at each month over the investment horizon for each simulation run. Then for each month, we average these values over the 10,000 simulation runs to compute average $\Theta_t$ and $\Phi_t$ values for each month over the investment horizon. Since in our example we elect monthly rebalancing, $\Phi_t$ is constant over the investment horizon.
Source: PGIM IAS. Example shown for illustrative purposes only.

Figure 15: Average Illiquid, Liquid Low-Risk and Liquid High-Risk Asset Portfolio Weights over Time
(10y horizon; 70% liquidation threshold; 90% confidence level)

Note: For a given optimal initial $(\Theta_0^*, \Phi_0^*)$ pair, we record the market value of the illiquid asset, the liquid low-risk asset, and the liquid high-risk asset, as a percentage of the portfolio’s total market value, in each month over the investment horizon for each simulation run. Then for each month, we average these values over the 10,000 simulation runs to compute the average market value weights for the three assets, for each month, over the investment horizon.
Source: PGIM IAS. Example shown for illustrative purposes only.
Figure 16: Distribution of Horizon Illiquid Asset Allocation \((1 - \Theta_T)\)

(10y horizon; 70% liquidation threshold; 90% confidence level; 5% illiquid return advantage)

Note: Assuming a 10y horizon, 70% liquidation threshold, 90% confidence level and 5% illiquid return advantage, we report the frequency of horizon portfolio allocation to the illiquid asset across the 10,000 simulation runs.

Source: PGIM IAS. Example shown for illustrative purposes only.

Figure 14 shows how the allocation to the liquid portfolio evolves \((\Theta_t)\), on average, over the 120 months (again assuming 70% liquidation threshold and 90% confidence level). The left panel is constructed by computing \(\Theta_t\) at each month, for each of the 10,000 simulation runs, and then averaging over the 10,000 runs. The figure shows that, on average, the weight of the illiquid asset gradually increases over time (i.e., the percentage of the liquid portfolio declines) from 19% at the beginning to 26% by the end of 120 months. This increase in allocation is driven by the illiquid asset having a higher expected return than the overall liquid portfolio.

The dynamics for the weight of liquid high-risk portfolio in the liquid portfolio \((\Phi_t)\) is shown in the right panel of Figure 14. Given that we elect monthly rebalancing of the liquid portfolio back to the optimal initial \({\Phi_0}^*\) weight, the value for \(\Phi_t\) is naturally constant over time.

Another way to view the portfolio’s asset allocation dynamics is to show the average percentage of all three assets, as a percentage of the total portfolio value, over time across all 10,000 simulation runs. Figure 15 shows that the allocation to the illiquid asset increases over time, and since we elect to rebalance the liquid portfolio, the weights of the liquid low-risk and liquid-high risk both decline pari passu even though the latter has a higher expected return.

While Figures 14 and 15 show the average allocation to the illiquid asset over time across all 10,000 simulation runs for a given optimal initial \({\Theta_0}^*, {\Phi_0}^*\) pair, investors may have an interest in the distribution of possible market values for the illiquid asset. Figure 16 shows a distribution of horizon \((1 - \Theta_T)\) values (i.e., percentage allocation to the illiquid asset), again assuming a 70% liquidation threshold and 90% confidence level. The average horizon allocation to the illiquid asset is 25.6% across the 10,000 simulation runs, and 80% of runs have an allocation between 14.6% and 37.6%.

**Summary**

Illiquid private assets offer investors opportunities not available in the liquid public markets: new sources of risk and return arising from the fundamental characteristics of these assets. Potentially higher expected returns constitute a “private market premium” over liquid public markets.

The higher expected returns of illiquid assets may help improve portfolio performance. However, a challenge for investors is to determine their allocation to illiquid assets subject to their cash flow obligations.

We present an asset allocation framework to help investors better understand the issues involved when allocating to illiquid assets. A feature of our framework is that we explicitly treat illiquid assets as illiquid – they cannot be sold until maturity. We assume the investor selects an asset allocation to maximize expected portfolio market value (or expected utility) at a horizon date, subject to the constraint
that the cash value of the liquid assets over the horizon is sufficient to satisfy any interim cash flow requirements at a pre-specified level of confidence. The risk for the investor is not having enough liquid assets over the investment horizon. Consequently, the fundamental trade-off for investors is between their portfolio value at the horizon date versus their confidence in being able to satisfy their liquidity needs up to the horizon date.

Our illiquid-liquid asset allocation framework addresses several issues that concern investors:

- How to determine the optimal allocation between liquid and illiquid assets based on the horizon portfolio value objective and liquidity requirements up to the horizon date. The framework highlights that this allocation can be quite sensitive to changes in the investor’s liquidity requirements, and the allocation may move in unanticipated directions at high liquidity requirements.

- The framework illustrates the interaction between an investor’s liquidity requirements and the investor’s objective of maximizing horizon portfolio value (i.e., returns). Specifically, the framework allows the investor to quantify the cost of changes to the investor’s liquidity constraint. For example, we show the degree to which portfolio expected returns decrease if the investor decides to be more confident in meeting the interim cash flow requirements (e.g., choosing to be 90% certain rather than 80% certain).

- We also show that the investor’s optimal illiquid-liquid asset allocation choice affects the optimal asset allocation within the investor’s liquid asset portfolio. For example, it may be optimal for the investor to hold a lower-risk liquid portfolio so that a larger allocation can be made to the illiquid asset.

- For investors already with an allocation to illiquid assets, the framework can be used to optimally structure the liquid portfolio.

Our suggested framework is intuitive and flexible, and can be customized to accommodate different asset types and liquidity requirements. Our hope is that this framework will help make institutional investors more comfortable investing in illiquid private assets.

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